The correspondence between Tracy-Widom (TW) and Adler-Shiota-van Moerbeke (ASvM) approaches in random matrix theory: the Gaussian case

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Abstract

Two approaches (TW and ASvM) to derivation of integrable differential equations for random matrix probabilities are compared. Both methods are rewritten in such a form that simple and explicit relations between all TW dependent variables and τ-functions of ASvM are found, for the example of finite size Gaussian matrices. Orthogonal function systems and Toda lattice are seen as the core structure of both approaches and their relationship.

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I. INTRODUCTION

Random matrices were introduced in the middle of last century by Wigner and Dyson as a description of complex systems that were difficult to study since they did not admit any exact or even approximate solutions. So the fact, thoroughly realized in the early nineties, is still amazing: many interesting random matrix (RM) models are closely related with and conveniently described by integrable systems — integrable hierarchies of partial differential equations (PDE). The first such result by Jimbo, Miwa, Môri and Sato[13] appeared earlier but remained unrecognized as an instance of a general rule for a decade. Matrix integrals, giving the probability distributions for eigenvalues of random matrices, proved to be τ-functions of integrable hierarchies. The ground-breaking works of Tracy and Widom[16,17] followed, which generalized and extended the results of Ref. 13 for the sine kernel to a number of other cases. The authors derived integrable PDE or Painlevé ODE as equations satisfied by the probabilities for the spectrum of various random matrices as well as by their large size asymptotics.

Another major approach was developed by Adler, Shiota and van Moerbeke[1−6]. It is based directly on the correspondence between matrix integrals over the spectral domains and τ-functions — the "generating functions" of integrable PDE. Using both methods, different integrable PDE have been obtained not only for single matrix ensembles but also for coupled Hermitian matrices[4,5,8,18]. In the latter case equations derived by different methods seem to be so dissimilar that the only indication of the existence of relations between them is that they are equations for the same probabilities.

In this paper, we compare the "algebraic" approach of Adler, Shiota and van Moerbeke with the "functional-theoretic" approach of Tracy and Widom and show the common structure behind them. This structure is (not surprisingly) the Toda lattice[19,1,3]. The relation between these two approaches has not been heretofore completely clarified. The reason lies in the fact that the authors of Refs. 2, 3 used the KP hierarchy rather than one-dimensional Toda lattice hierarchy (1-Toda) to derive their PDE for single Hermitian matrices. This led to equations very different in form from those derived in Refs. 16, 17. J.Harnad[11] demonstrated for the special cases of Airy and Bessel kernels, how the "KP"-equation of ASvM can be obtained by taking the appropriate combinations of TW equations. We show that there is a more direct and complete correspondence of structures involving the Toda lattice hierarchy; and thus the procedure of Ref. 11 is essentially an implicit transition from 1-Toda to KP form of integrable equations related to Hermitian 1-matrix models. In this paper we restrict ourselves to the Hermitian one-matrix Gaussian case. Our main purpose is to reveal explicitly the underlying structures connecting dependent variables and equations in the different approaches. As we show by this example, for single matrices equations obtained by different methods (Ref. 17 vs. Refs. 2, 3) can essentially coincide, despite the very different ways to arise. For comparison, we give the implementation of the Harnad's approach in the Appendix for the case of finite size Gaussian matrices considered here.

In another related work[10], 1-Toda lattice was ingeniously used to treat the Virasoro constraints of ASvM approach, which, together with an equation of Ref. 17, helped to find Painlevé VI equation for the Jacobi ensemble.

II. SUMMARY OF RESULTS

We use the ASvM approach of Hirota's bilinear identities for τ-functions of the 1-dimensional Toda lattice hierarchy[19] and (linear) Virasoro constraints. Here the τ-function τ_n^J has the meaning of complex Hermitian Gaussian matrix integral over spectral domain J^n, $J = \cup_i(a_{2i-1}, a_{2i})$ – a finite union of intervals of the real axis $\mathbb{R}$ with endpoints a_i:

$$\tau_n^J(t) = \frac{1}{n!} \int_{J^n} \prod_{i<j}(x_i - x_j)^2 \prod_1^n \rho_t(x_i)dx_i,$$

where n is the random matrix size, $t = (t_1, t_2, t_3, ...)$ is a complex infinite vector, $\rho_t(x) = e^{-V(x)+\sum_{k=1}^{\infty} t_k x^k}$ – the t-deformed weight; and $\tau_n(t)$ will denote the corresponding integral over the whole $\mathbb{R}^n$. Our first results are the equations

$$\mathcal{B}_{-1}^2 \ln \tau_n^J = 2(2Q_n P_n - n), \tag{1}$$

$$\mathcal{B}_{-1}^2 Q_n = (-2\mathcal{B}_0 + 4n)Q_n - 8Q_n^2 P_n, \tag{2}$$

$$\mathcal{B}_{-1}^2 P_n = (2\mathcal{B}_0 + 4n)P_n - 8Q_n P_n^2, \tag{3}$$

where $Q_n = \tau_{n+1}^J/\tau_n^J$, $P_n = \tau_{n-1}^J/\tau_n^J$, and the operators $\mathcal{B}_k$ contain partial derivatives w.r.t. the boundary points of the intervals:

$$\mathcal{B}_k \tau_n^J = \sum_{i\in\{endpoints\}} a_i^{k+1} \frac{\partial}{\partial a_i} \tau_n^J.$$

Then, analyzing the TW equations for the finite n Hermite case, we obtain simple relations among the above τ-functions and their ratios and the auxiliary functions u, v, w, appearing in Ref. 17. The TW approach consists in studying the kernel of a Fredholm integral operator K_n^J, whose determinant is the probability of having no eigenvalues in a subset of $\mathbb{R}$, here – the complement J^c of the set J:

$$\det(I - K_n^J) = \frac{\tau_n^J}{\tau_n}. \tag{4}$$

where the numerator and denominator of the r.h.s. are the matrix integrals (the τ functions $\tau(t=0)$) over J and over real axis $\mathbb{R}$, respectively. The operator K_n^J has kernel

$$K_n^J(x, y) = K_n(x, y)\chi_{J^c}(y), \tag{5}$$

where $\chi_{J^c}(x)$ is the characteristic function of J^c:

$$\chi_{J^c}(x) = \begin{cases} 1, & if \ x \in J^c \\ 0, & if \ x \notin J^c \end{cases},$$

$K_n(x, y)$ is an integrable kernel[12], which can be expressed by Christoffel-Darboux formula for orthogonal functions,

$$K_n(x, y) = \frac{\phi(x)\psi(y) - \psi(x)\phi(y)}{x - y} = \sum_{k=0}^{n-1} \phi_k(x)\phi_k(y), \tag{6}$$

$\phi(x) = \sqrt{b_{n-1}}\phi_n(x)$, $\psi(x) = \sqrt{b_{n-1}}\phi_{n-1}(x)$, and b_{n-1} is the coefficient of $\phi_{n-1}(x)$ in the corresponding 3-term recurrence relation.

We show that the inner product $v = (\phi, (I - K_n^J)^{-1}\psi)$ is related to τ_n^J in the following way:

$$v \equiv (\phi, (I - K_n^J)^{-1}\psi) = \frac{\mathcal{B}_{-1}\ln\tau_n^J}{2}. \tag{7}$$

Also, if we denote

$$\tilde{Q}_n = \sqrt{n/2} - u, \qquad \tilde{P}_n = w + \sqrt{n/2}, \tag{8}$$

where $u = (\phi, (I - K_n^J)^{-1}\phi)$, $w = (\psi, (I - K_n^J)^{-1}\psi)$, then these functions $\tilde{Q}_n$ and $\tilde{P}_n$ turn out to satisfy the same nonlinear PDE (2) and (3) as the ratios of τ-functions Q_n and P_n, respectively. These facts lead to even stronger conclusions:

$$\tilde{Q}_n = C_n Q_n, \qquad \tilde{P}_n = Q_n/C_n,$$

where $C_n = (2^n/n!) \cdot \sqrt{n/2\pi}$. So the functions u and w appearing in the resolvent kernel approach, are represented in terms of ratios of matrix integrals (τ-functions at $t = 0$):

$$u \equiv (\phi, (I - K_n^J)^{-1}\phi) = \sqrt{\frac{n}{2}}\left(1 - \frac{2^n}{n!\sqrt{\pi}}\frac{\tau_{n+1}^J}{\tau_n^J}\right) = \sqrt{\frac{\tau_{n+1}\tau_{n-1}}{(\tau_n)^2}}\left(1 - \frac{\tau_{n+1}^J/\tau_{n+1}}{\tau_n^J/\tau_n}\right), \tag{9}$$

$$w \equiv (\psi, (I - K_n^J)^{-1}\psi) = \sqrt{\frac{n}{2}}\left(\frac{(n-1)!\sqrt{\pi}}{2^{n-1}}\frac{\tau_{n-1}^J}{\tau_n^J} - 1\right) = \sqrt{\frac{\tau_{n+1}\tau_{n-1}}{(\tau_n)^2}}\left(\frac{\tau_{n-1}^J/\tau_{n-1}}{\tau_n^J/\tau_n} - 1\right). \tag{10}$$

Thus, the correspondence between TW auxiliary variables and τ-functions of one-dimensional Toda lattice hierarchy considered by Adler and van Moerbeke,[1,3] becomes direct and explicit. The *probabilities*, expressed in the form of τ-ratios are seen from the above formulas to be also the natural and convenient auxiliary coordinates. Indeed, τ-functions and their ratios seem to provide the best description for all integrable problems. Here, in the Gaussian case, for instance, we have the following recurrence relations for the τ-ratios:

$$P_{n+1} = \frac{1}{Q_n}, \tag{11}$$

$$Q_{n+1} = Q_n\left(\frac{\mathcal{B}_{-1}^2\ln Q_n}{4} + Q_n P_n + \frac{1}{2}\right), \tag{12}$$

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the first — by definition of Q_n, P_n, the second follows from Eq. (1) and the first one. Thus in principle these ratios can be recursively determined from the simplest ones (Q_0 and Q_1 here, with our normalization, $Q_0 = \tau_1^J = \int_J e^{-x^2} dx$, $Q_1 = \tau_2^J/\tau_1^J$, $\tau_2^J = 1/2 \int_{J^2} e^{-x^2-y^2}(x-y)^2 dx dy$).

Remark. The 1-Toda lattice τ-functions are naturally related to the orthogonal polynomial structure and this is the reason for the existence of such simple relations. The orthogonal functions for t-deformed weight satisfy the t-dependent three-term recurrence relation

$$x\phi_n(x;t) = a_n(t)\phi_n(x;t) + b_n(t)\phi_{n+1}(x;t) + b_{n-1}(t)\phi_{n-1}(x;t),$$

where the coefficients depend on times and are known in terms of the Toda τ-functions[1,3]:

$$a_n(t) = \frac{\partial}{\partial t_1} \ln \frac{\tau_{n+1}}{\tau_n},$$

$$b_{n-1}^2(t) = \frac{\tau_{n+1}\tau_{n-1}}{\tau_n^2}.$$

These relations show that the 1-Toda time flows are in fact flows between different *orthogonal function systems*. The 1-Toda bilinear relations may also be considered as a form of orthogonality relations for systems of orthogonal functions. Recall also that Christoffel-Darboux formula, which is the starting point of the resolvent kernel approach, is itself equivalent to the three-term relations for the corresponding orthogonal functions. One can see that the structure of orthogonal function systems is at the heart of all integrability derivations and connections with random matrix theory.

III. DERIVATION OF EQUATIONS (1),(2),(3): BILINEAR IDENTITIES AND VIRASORO CONSTRAINTS

We consider Hermitian random matrices of size $n \times n$ with spectral measure $d\mu = \rho(x)dx$, $\rho(x) = e^{-V(x)}$ such that all its moments exist. Then the probability that all eigenvalues are in $J \subseteq \mathbb{R}$ is

$$\mathbb{P}(all \ x_i \in J) = \frac{\int_{J^n} \Delta_n^2(x) \prod_1^n \rho(x_i)dx_i}{\int_{\mathbb{R}^n} \Delta_n^2(x) \prod_1^n \rho(x_i)dx_i}, \tag{13}$$

where $\Delta_n(x) = \prod_{i<j}(x_i - x_j)$ is the Vandermonde determinant. The τ-function of integrable hierarchy appears from deformation of the above integrals: let $t = (t_1, t_2, t_3, ...)$ be a complex infinite vector of "times" and $\rho_t(x) = e^{-V(x)+\sum_{k=1}^{\infty} t_k x^k}$ – the deformed weight, then the matrix integral

$$\tau_n^J(t) = \frac{1}{n!} \int_{J^n} \Delta_n^2(x) \prod_1^n \rho_t(x_i)dx_i, \tag{14}$$

is a τ-function of integrable hierarchies such as KP and one-dimensional Toda lattice hierarchy (see, for instance, Refs. $1-3$).

Matrix integrals for coupled random matrices satisfy[4] Hirota bilinear identities for 2-dimensional Toda lattice (2-Toda)[19]:

$$\oint_{z=\infty} \tau_n(t - [z^{-1}], s)\tau_{m+1}(t' + [z^{-1}], s')e^{\sum_1^\infty (t_k - t'_k)z^k} z^{n-m-1}dz =$$

$$= \oint_{z=0} \tau_{n+1}(t, s - [z])\tau_m(t', s' + [z])e^{\sum_1^\infty (s_k - s'_k)z^{-k}} z^{n-m-1}dz \tag{15}$$

The 2-Toda τ-functions have two infinite sets of time parameters. There are two possible reductions of them to the bilinear identities for 1-matrix integral τ-functions, possessing only one set of times (first was written in Ref. 3, second – in Ref. 6). The first one can be obtained by setting $\tau(t, s) = \tau(t - s)$ and $s - s' = -(t - t')$, then renaming variables $t - s \to t$, $t' - s' \to t'$. It gives the bilinear identities[3] for 1-dimensional Toda lattice hierarchy (1-Toda) (including the AKNS hierarchy and the one-dimensional Toda chain itself, see, e.g., Ref. 19):

$$\oint_{z=\infty} \tau_n(t - [z^{-1}])\tau_{m+1}(t' + [z^{-1}])e^{1/2 \sum_1^\infty (t_k - t'_k)z^k} z^{n-m-1}dz =$$

$$= \oint_{z=0} \tau_{n+1}(t + [z])\tau_m(t' - [z])e^{-1/2 \sum_1^\infty (t_k - t'_k)z^{-k}} z^{n-m-1}dz \tag{16}$$

The second one is obtained by setting $\tau(t, s) = \tau(t - s)$ and $s = s'$, then again renaming $t - s \to t$, $t' - s' \to t'$. This gives the bilinear identities[6] for the *discrete* KP hierarchy for $n > m$:

$$\oint_{z=\infty} \tau_n(t - [z^{-1}])\tau_{m+1}(t' + [z^{-1}])e^{\sum_1^\infty (t_k - t'_k)z^k} z^{n-m-1}dz =$$

$$= \oint_{z=0} \tau_{n+1}(t + [z])\tau_m(t' - [z])z^{n-m-1}dz = 0 \ for \ n > m \tag{17}$$

The most familiar bilinear identity of the usual KP hierarchy[9] corresponds to taking $m = n-1$ in (17). The integrable nonlinear PDE can be readily obtained from the above identities by shifting times $t \to t + a$, $t' \to t - a$, expanding in Taylor series in a and taking residues for each term of this expansion independently.

Let us proceed in the 1-Toda setting, which, as we will see, shows most clearly the correspondence between the ASvM and the TW approaches to the derivation of nonlinear integrable PDE for the spectral probabilities of random matrices.

The simplest and most important equations are obtained from (16) for $m = n$ and $m = n - 1$ upon taking the residues of the terms linear in $2a = t - t'$. For $m = n$ one gets the hierarchy of equations in terms of τ-functions

$$\frac{\partial}{\partial t_k} \ln \frac{\tau_{n+1}}{\tau_n} = \sum_{i=0}^k \frac{p_i(-\tilde\partial_t)\tau_n \cdot p_{k-i}(\tilde\partial_t)\tau_{n+1}}{\tau_n \tau_{n+1}}, \tag{18}$$

where $p_k(t)$ are the elementary Schur polynomials defined by

$$e^{\sum_1^\infty t_k z^k} = \sum_{k=0}^\infty p_k(t) z^k, \tag{19}$$

$\tilde{\partial}_t = (\partial_{t_1}, 1/2\partial_{t_2}, 1/3\partial_{t_3}, ...)$, so that one can write

$$\tau_n(t \mp [z^{-1}]) = \sum_{k=0}^\infty z^{-k} p_k(\mp\tilde{\partial}_t)\tau_n(t). \tag{20}$$

Equation (18) is trivial when $k = 1$. The simplest nontrivial equation arises from (18) when $k = 2$. It can be written as

$$\frac{\partial}{\partial t_2} \ln \frac{\tau_{n+1}}{\tau_n} = \frac{\partial^2}{\partial t_1^2} \ln \frac{\tau_{n+1}}{\tau_n} + \left(\frac{\partial}{\partial t_1} \ln \frac{\tau_{n+1}}{\tau_n}\right)^2 + 2\frac{\partial^2 \ln \tau_n}{\partial t_1^2}, \tag{21a}$$

or, introducing the new variable $Q_n = \tau_{n+1}/\tau_n$, as

$$\frac{\partial Q_n}{\partial t_2} = \frac{\partial^2 Q_n}{\partial t_1^2} + 2\frac{\partial^2 \ln \tau_n}{\partial t_1^2} Q_n. \tag{21b}$$

Now consider the 1-Toda bilinear identity for $m = n - 1$. The same procedure of taking residue for terms of order a as above now gives another infinite set of integrable PDE:

$$2\frac{\partial^2}{\partial t_1 \partial t_k} \ln \tau_n - \sum_{i=0}^{k+1} \frac{p_i(-\tilde{\partial}_t)\tau_n \cdot p_{k+1-i}(\tilde{\partial}_t)\tau_n}{\tau_n^2} = \sum_{i=0}^{k-1} \frac{p_i(-\tilde{\partial}_t)\tau_{n+1} \cdot p_{k-1-i}(\tilde{\partial}_t)\tau_{n-1}}{\tau_n^2}.$$

Then the simplest nontrivial equation is obtained when $k = 1$, and this is the standard Toda equation for 1-dimensional Toda chain expressed in terms of τ-functions:

$$\frac{\partial^2 \ln \tau_n}{\partial t_1^2} = \frac{\tau_{n+1}\tau_{n-1}}{\tau_n^2}. \tag{22}$$

Thus we obtained a closed system of three nonlinear integrable equations for three dependent variables, $\ln \tau_n$, Q_n and P_n:

$$\frac{\partial^2 \ln \tau_n}{\partial t_1^2} = Q_n P_n, \tag{23}$$

$$\frac{\partial Q_n}{\partial t_2} = \frac{\partial^2 Q_n}{\partial t_1^2} + 2\frac{\partial^2 \ln \tau_n}{\partial t_1^2} Q_n, \tag{24}$$

$$-\frac{\partial P_n}{\partial t_2} = \frac{\partial^2 P_n}{\partial t_1^2} + 2\frac{\partial^2 \ln \tau_n}{\partial t_1^2} P_n, \tag{25}$$

which is nothing but the well-known Toda-AKNS system (see, for instance, Ref. 15).

Next we will use the well-known Virasoro constraints to express the time derivatives in the previous equations in terms of *boundary* derivatives, i.e. the derivatives with respect

to the endpoints of the intervals where the matrix eigenvalues lie. To make the paper self-contained, we reproduce here the derivation of Virasoro constraints for the general β one-matrix ensemble.[7,4,3] Although for the purposes of current paper we will need only the simplest case $\beta = 2$ corresponding to Hermitian ensemble, it is instructive and not more difficult to give the derivation for arbitrary real nonnegative β. The Virasoro constraints arise from the obvious invariance of the matrix integral (the τ-function) under an arbitrary change of integration variables. Consider infinitesimal such changes of the form $x_i \to x_i + \varepsilon x_i^{k+1}$, $\varepsilon \to 0$, $k \geq -1$. Then the generators $\mathcal{V}_k$ of such changes of coordinates make a subalgebra of Virasoro algebra. The change in the integrand, which will be expressed in terms of the time derivatives of the integral, is compensated by the corresponding change of the limits of integration, which in turn will be expressed in terms of the boundary derivatives of the integral. Consider the change in the integrand dI_n of the matrix integral,[7,4,3]

$$
\left. \frac{d(dI_n(x + \varepsilon x^{k+1}))}{d\varepsilon} \right|_{\varepsilon=0}, \qquad \int_{J^n} dI_n(x) = \tau_n^J(t) = \frac{1}{n!} \int_{J^n} \Delta_n^\beta(x) \prod_1^n \rho_t(x_i) dx_i. \tag{14$'$}
$$

If $k > 0$ the change of the Vandermonde determinant Δ_n is

$$
\left. \frac{d\Delta_n^\beta(x + \varepsilon x^{k+1})}{d\varepsilon} \right|_{\varepsilon=0} = \beta \Delta_n^{\beta-1}(x) \left. \frac{d\Delta_n(x + \varepsilon x^{k+1})}{d\varepsilon} \right|_{\varepsilon=0} = \beta \Delta_n^\beta(x) \sum_{i=1}^n \sum_{j>i}^n \frac{x_i^{k+1} - x_j^{k+1}}{x_i - x_j} =
$$

$$
= \Delta_n^\beta(x) \left(\frac{1}{2} \sum_{k_1+k_2=k, k_l>0} \sum_{i,j=1}^n x_i^{k_1} x_j^{k_2} + \left(n - \frac{k+1}{2} \right) \sum_{i=1}^n x_i^k \right),
$$

also

$$
\prod_{i=1}^n \left. \frac{d(d(x_i + \varepsilon x_i^{k+1}))}{d\varepsilon} \right|_{\varepsilon=0} = (k+1) \sum_{i=1}^n x_i^k \cdot \prod_{i+1}^n dx_i.
$$

Using the identity

$$
\frac{\partial}{\partial t_k} dI_n = \sum_{i=1}^n x_i^k \cdot dI_n, \tag{26}
$$

one gets (the second term on the r.h.s. is absent for $k = 1$)

$$
\left. \frac{d(dI_n(x + \varepsilon x^{k+1}))}{d\varepsilon} \right|_{\varepsilon=0} =
$$

$$
= \left\{ -\sum_{i=1}^n x_i^{k+1} \frac{dV}{dx_i} + \frac{\beta}{2} \sum_{l=1}^{k-1} \frac{\partial^2}{\partial t_l \partial t_{k-l}} + \sum_{l=1}^\infty l t_l \frac{\partial}{\partial t_{l+k}} + [\beta n + (1 - \beta/2)(k+1)] \frac{\partial}{\partial t_k} \right\} dI_n.
$$

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For $k = -1$ the measure $\Delta_n^\beta(x) \prod_1^n dx_i$ does not change under translation $x \to x + \varepsilon$, so one has instead

$$\left. \frac{d(dI_n(x + \varepsilon))}{d\varepsilon} \right|_{\varepsilon=0} = \left\{ -\sum_{i=1}^n \frac{dV}{dx_i} + nt_1 + \sum_{l=2}^\infty lt_l \frac{\partial}{\partial t_{l-1}} \right\} dI_n.$$

For $k = 0$ one has for the change of the measure

$$\left. \frac{d(\Delta_n^\beta \prod_i dx_i(x + \varepsilon x))}{d\varepsilon} \right|_{\varepsilon=0} = \left(\frac{\beta}{2} n(n-1) + n \right) \Delta_n^\beta(x) \prod_i dx_i,$$

and one gets

$$\left. \frac{d(dI_n(x + \varepsilon x))}{d\varepsilon} \right|_{\varepsilon=0} = \left\{ -\sum_{i=1}^n x_i \frac{dV}{dx_i} + \sum_{l=1}^\infty lt_l \frac{\partial}{\partial t_l} + n + \frac{\beta}{2} n(n-1) \right\} dI_n.$$

Thus one finally obtains the $k = -1$ constraint:

$$\mathcal{B}_{-1} \tau_n^J = \mathcal{V}_{-1} \tau_n^J = \int \left\{ -\sum_{i=1}^n \frac{dV}{dx_i} + nt_1 + \sum_{l=2}^\infty lt_l \frac{\partial}{\partial t_{l-1}} \right\} dI_n(x), \qquad (27a)$$

the $k = 0$ constraint:

$$\mathcal{B}_0 \tau_n^J = \mathcal{V}_0 \tau_n^J = \int \left\{ -\sum_{i=1}^n x_i \frac{dV}{dx_i} + n + \beta \frac{n(n-1)}{2} + \sum_{l=1}^\infty lt_l \frac{\partial}{\partial t_l} \right\} dI_n(x), \qquad (27b)$$

and the $k > 0$ constraints (again the second term on the r.h.s. is absent for $k = 1$):

$$\mathcal{B}_k \tau_n^J = \mathcal{V}_k \tau_n^J =$$

$$\int \left\{ -\sum_{i=1}^n x_i^{k+1} \frac{dV}{dx_i} + \frac{\beta}{2} \sum_{l=1}^{k-1} \frac{\partial^2}{\partial t_l \partial t_{k-l}} + [\beta n + (1 - \beta/2)(k+1)] \frac{\partial}{\partial t_k} + \sum_{l=1}^\infty lt_l \frac{\partial}{\partial t_{l+k}} \right\} dI_n(x).$$
$$(27c)$$

If the potential $V(x)$ is polynomial, $\frac{dV}{dx} = \sum_{l=0} g_l x^l$, then one has

$$\sum_{i=1}^n x_i^{k+1} \frac{dV}{dx_i} = \sum_{l=0} g_l \frac{\partial}{\partial t_{l+k+1}}.$$

The infinitesimal changes of integration limits are expressed in terms of boundary operators[1−3]

$$\mathcal{B}_k \tau_n^J = \sum_{i \in \{endpoints\}} a_i^{k+1} \frac{\partial}{\partial a_i} \tau_n^J. \qquad (28)$$

In general, also the next important consequences of the above Virasoro constraints are needed[1−3]:

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$$\mathcal{B}_k \mathcal{B}_l \tau_n^J = \mathcal{B}_k \mathcal{V}_l \tau_n^J = \mathcal{V}_l \mathcal{B}_k \tau_n^J = \mathcal{V}_l \mathcal{V}_k \tau_n^J, \tag{29}$$

the second equality holds because the operators $\mathcal{B}_k$ and $\mathcal{V}_l$ commute.

From now on we consider the special case of matrices with Gaussian distribution of matrix elements, i.e. Gaussian potential $V(x) = x^2$. At the locus $t_1 = t_2 = \ldots = 0$, the Virasoro constraints (27a), (27b) together with identity (26) give in this case[2,3]:

$$\frac{\partial \ln \tau_n^J}{\partial t_1} = -\frac{\mathcal{B}_{-1} \ln \tau_n^J}{2}, \tag{30}$$

$$\frac{\partial^2 \ln \tau_n^J}{\partial t_1^2} = \frac{\mathcal{B}_{-1}^2 \ln \tau_n^J}{4} + \frac{n}{2}, \tag{31}$$

$$\frac{\partial \ln \tau_n^J}{\partial t_2} = -\frac{\mathcal{B}_0 \ln \tau_n^J}{2} + \frac{n^2}{2}. \tag{32}$$

These are the only Virasoro constraints we will need here. More such relations (involving also the use of (27c) for $k = 1$) are needed for the case of KP-hierarchy,[2,3] and the derivation is longer. We substitute the relations (30)–(32) into (23), (24), (25) and after a simple algebra obtain the "boundary form" of equations (23)–(25), which is our system (1)–(3).

If we used the KP hierarchy to derive the PDE as Adler, Shiota and Van Moerbeke did in Refs. $2, 3$, rather than 1-Toda, we would have used just one standard integrable PDE — the KP equation,

$$\frac{\partial^4 \ln \tau_n}{\partial t_1^4} + 3\frac{\partial^2 \ln \tau_n}{\partial t_2^2} - 4\frac{\partial^2 \ln \tau_n}{\partial t_1 \partial t_3} + 6\left(\frac{\partial^2 \ln \tau_n}{\partial t_1^2}\right)^2 = 0. \tag{33}$$

Then we would need more Virasoro constraints like eqs. (30)–(32) (involving also the third time t_3, the second derivative w.r.t. t_2 and the fourth derivative w.r.t. t_1). Then the ASvM boundary KP equation would come out, for the finite n Gaussian case it reads[2]:

$$(\mathcal{B}_{-1}^4 + 8n\mathcal{B}_{-1}^2 + 12\mathcal{B}_0^2 + 24\mathcal{B}_0 - 16\mathcal{B}_{-1}\mathcal{B}_1) \ln \tau_n^J + 6(\mathcal{B}_{-1}^2 \ln \tau_n^J)^2 = 0. \tag{34}$$

IV. TRACY-WIDOM EQUATIONS AND τ-FUNCTIONS: THE GAUSSIAN CASE

We consider the same Hermitian Gaussian random matrices as above. Various large n limits for this case belong to important widespread universality classes: bulk limit leads to equations for the Sine kernel,[13,17] whereas edge limit (largest eigenvalues) leads to Airy kernel.[16,17] One studies the operator K_n^J (4) with the kernel (5), (6) and $b_{n-1} = \sqrt{n/2}$ for the Hermite case. Also important for this theory is the resolvent kernel of K (we will use short-hand notation K for K_n^J), $R(x,y)$, the kernel of $K(I-K)^{-1}$, which is defined by the operator identity

$$(I + R)(I - K) = I. \tag{35}$$

Then one introduces[17] the auxiliary functions (denoted in Ref. 17 as Q and P, respectively)

$$q(x; J) = (I - K)^{-1}\phi(x), \qquad p(x; J) = (I - K)^{-1}\psi(x), \tag{36}$$

and auxiliary inner products, which are functions of only the endpoints a_k of J,

$$u = (q, \phi\chi_{J^c}), \qquad v = (q, \psi\chi_{J^c}) = (p, \phi\chi_{J^c}), \qquad w = (p, \psi\chi_{J^c}). \tag{37}$$

Then the resolvent kernel $R(x, y)$ is[12,17]

$$R(x, y) = \frac{q(x; J)p(y; J) - p(x; J)q(y; J)}{x - y}, \quad (x, y \in J^c, \ x \neq y) \tag{38}$$

and $R(x, x) = p(x; J)q'(x; J) - q(x; J)p'(x; J)$. The TW equations for 1-matrix case are the equations for the functions of the endpoints — $R(a_j, a_k)$,

$$q_k = q(a_k; J), \qquad p_k = p(a_k; J),$$

and the auxiliary functions u, v, w. Among them there are universal equations valid for any Hermitian 1-matrix unitarily invariant model. They read[17]

$$\frac{\partial q_j}{\partial a_k} = (-1)^k R(a_j, a_k)q_k, \tag{39}$$

$$\frac{\partial p_j}{\partial a_k} = (-1)^k R(a_j, a_k)p_k, \tag{40}$$

$$R_{jk} \equiv R(a_j, a_k) = \frac{q_j p_k - p_j q_k}{a_j - a_k}, \tag{41}$$

for $j \neq k$ and

$$R_k \equiv R(a_k, a_k) = p_k \frac{\partial q_k}{\partial a_k} - q_k \frac{\partial p_k}{\partial a_k}, \tag{42}$$

$$\frac{\partial u}{\partial a_k} = (-1)^k q_k^2, \tag{43}$$

$$\frac{\partial v}{\partial a_k} = (-1)^k q_k p_k, \tag{44}$$

$$\frac{\partial w}{\partial a_k} = (-1)^k p_k^2. \tag{45}$$

The other equations are not universal, their particular form depends on the potential $V(x)$. For the finite n Gaussian (Hermite) case they are[17]

$$\frac{\partial q_j}{\partial a_j} = -a_j q_j + (\sqrt{2n} - 2u)p_j - \sum_{k \neq j}(-1)^k R(a_j, a_k)q_k, \tag{46}$$

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$$\frac{\partial p_j}{\partial a_j} = a_j p_j - (\sqrt{2n} + 2w)q_j - \sum_{k \neq j}(-1)^k R(a_j, a_k)p_k, \tag{47}$$

along with

$$R(a_j, a_j) = -2a_j q_j p_j + (\sqrt{2n} - 2u)p_j^2 + (\sqrt{2n} + 2w)q_j^2 + \sum_{k \neq j}(-1)^k R(a_j, a_k)(q_j p_k - p_j q_k), \tag{48}$$

$$\frac{\partial R(a_j, a_j)}{\partial a_j} = -2q_j p_j - \sum_{k \neq j}(-1)^k R(a_j, a_k)^2. \tag{49}$$

It follows from (39)–(41), (46) and (47) that

$$\sum_k \frac{\partial}{\partial a_k} q_j p_j = (\sqrt{2n} - 2u)p_j^2 - (\sqrt{2n} + 2w)q_j^2,$$

and the last equation together with (43), (45) gives the first integral obtained in Ref. 17:

$$2\sum_j (-1)^j q_j p_j = -(2u - \sqrt{2n})(2w + \sqrt{2n}) - 2n,$$

or, if we denote $\sqrt{2n} - 2u = 2\tilde{Q}_n$, $2w + \sqrt{2n} = 2\tilde{P}_n$, as in formula (8), we get

$$\sum_j (-1)^j q_j p_j = 2\tilde{Q}_n \tilde{P}_n - n, \tag{50}$$

From (39)–(41), (46) and (47) it follows also that

$$\sum_k \frac{\partial q_j}{\partial a_k} = -a_j q_j + (\sqrt{2n} - 2u)p_j, \tag{51}$$

$$\sum_k \frac{\partial p_j}{\partial a_k} = a_j p_j - (\sqrt{2n} + 2w)q_j. \tag{52}$$

Then consider the expression (using (43), (51), (8) and (50))

$$\mathcal{B}_{-1}^2 \tilde{Q}_n = -\sum_{l,k} \frac{\partial}{\partial a_l} \frac{\partial}{\partial a_k} u = -2\sum_k (-1)^k q_k \left(\sum_l \frac{\partial q_k}{\partial a_l} \right) = 2\sum_k (-1)^k q_k (a_k q_k - 2\tilde{Q}_n p_k)$$

$$= 2\left(\sum_k a_k(-1)^k q_k^2 - 2\tilde{Q}_n \sum_k (-1)^k q_k p_k \right) = -2\sum_k a_k \frac{\partial \tilde{Q}_n}{\partial a_k} - 4\tilde{Q}_n(2\tilde{Q}_n \tilde{P}_n - n),$$

which is nothing but

$$\mathcal{B}_{-1}^2 \tilde{Q}_n = 2(-\mathcal{B}_0 + 2n)\tilde{Q}_n - 8\tilde{Q}_n^2 \tilde{P}_n, \tag{2'}$$

in terms of the earlier introduced $\tilde{Q}_n$. Completely analogously one can obtain

$$\mathcal{B}_{-1}^2 \tilde{P}_n = 2(\mathcal{B}_0 + 2n)\tilde{P}_n - 8\tilde{Q}_n \tilde{P}_n^2, \tag{3'}$$

in terms of $\tilde{P}_n$ introduced in (8).

Now let us prove the last equation

$$\mathcal{B}_{-1}^2 \ln \tau_n^J = 2(2\tilde{Q}_n \tilde{P}_n - n) \tag{1'}$$

with the same $\tilde{Q}_n$ and $\tilde{P}_n$ defined by (8). Recall formula (4) for the Fredholm determinant as a ratio of τ-functions. On the one hand, it follows from (4) that

$$\mathcal{B}_{-1} \ln \tau_n^J = \sum_k \frac{\partial}{\partial a_k} \ln \tau_n^J = \sum_k \frac{\partial}{\partial a_k} \ln \det(I - K_n^J), \tag{53}$$

on the other hand,

$$\sum_k \frac{\partial}{\partial a_k} \ln \det(I - K_n^J) = \sum_k \frac{\partial}{\partial a_k} Tr \ln(I - K_n^J) =$$

$$= Tr(I - K_n^J)^{-1} \sum_k \frac{\partial}{\partial a_k} K_n(x,y)\chi_{J^c}(y;a) = -\sum_k (-1)^k R_k. \tag{54}$$

Thus, we have

$$\mathcal{B}_{-1} \ln \tau_n^J = -\sum_k (-1)^k R_k. \tag{55}$$

Taking expression (48) for $R_k \equiv R(a_k, a_k)$ we get

$$\sum_k (-1)^k R_k = -2 \sum_k (-1)^k (a_k q_k p_k - \tilde{Q}_n p_k^2 - \tilde{P}_n q_k^2) - \sum_{j,k \neq j} (-1)^{j+k} R_{jk}(p_j q_k - q_j p_k),$$

the last double sum above, however, is equal to zero. By (43), (45), the last expression is

$$\sum_k (-1)^k R_k = -2 \sum_k (-1)^k a_k q_k p_k + 2\tilde{Q}_n \sum_k \frac{\partial \tilde{P}_n}{\partial a_k} - 2\tilde{P}_n \sum_k \frac{\partial \tilde{Q}_n}{\partial a_k}. \tag{56}$$

Another auxiliary formula we need follows from (51), (52), (43) and (45):

$$\sum_{k,j} (-1)^k a_k \frac{\partial(q_k p_k)}{\partial a_j} = 2\sum_k (-1)^k a_k(\tilde{Q}_n p_k^2 - \tilde{P}_n q_k^2) = 2\sum_k a_k \left(\tilde{Q}_n \frac{\partial \tilde{P}_n}{\partial a_k} + \tilde{P}_n \frac{\partial \tilde{Q}_n}{\partial a_k}\right) = 2\mathcal{B}_0(\tilde{Q}_n \tilde{P}_n). \tag{57}$$

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Using (56),(57) and (50), we derive from (55):

$$\mathcal{B}_{-1}^2 \ln \tau_n^J = 2\sum_k (-1)^k a_k \sum_j \frac{\partial(q_k p_k)}{\partial a_j} + 2\sum_k (-1)^k q_k p_k - 2\tilde{Q}_n \mathcal{B}_{-1}^2 \tilde{P}_n + 2\tilde{P}_n \mathcal{B}_{-1}^2 \tilde{Q}_n =$$

$$= 4\mathcal{B}_0(\tilde{Q}_n \tilde{P}_n) + 4\tilde{Q}_n \tilde{P}_n - 2n - 2\tilde{Q}_n \mathcal{B}_{-1}^2 \tilde{P}_n + 2\tilde{P}_n \mathcal{B}_{-1}^2 \tilde{Q}_n,$$

and, using (2'), (3'), this is exactly

$$\mathcal{B}_{-1}^2 \ln \tau_n^J = 2(2\tilde{Q}_n \tilde{P}_n - n).$$

Equation (1') with TW variables is proven. Thus, we have shown that Toda lattice stands behind TW derivations.

V. TW VARIABLES IN TERMS OF τ-FUNCTIONS

The TW u and w variables for Gaussian case turn out to be directly related to the ratios of Toda lattice τ-functions τ_{n+1}/τ_n and τ_{n-1}/τ_n. Moreover, if we compare formulas (1'), (44) and (50), we see that in the Gaussian case, the equation holds:

$$\mathcal{B}_{-1}^2 \ln \tau_n^J = 2(2\tilde{Q}_n \tilde{P}_n - n) = 2\mathcal{B}_{-1}v, \tag{58}$$

or

$$\mathcal{B}_{-1}(\mathcal{B}_{-1} \ln \tau_n^J - 2v) = 0. \tag{59}$$

Besides, the equations (58) and (55), (56) mean, respectively, also the following relations among τ-functions and TW variables in the Gaussian case:

$$Q_n P_n = \tilde{Q}_n \tilde{P}_n, \tag{60}$$

$$\mathcal{B}_{-1} \ln \tau_n^J = 2\mathcal{B}_0 v + 2\tilde{P}_n \mathcal{B}_{-1}\tilde{Q}_n - 2\tilde{Q}_n \mathcal{B}_{-1}\tilde{P}_n. \tag{61}$$

Then from (2), (3) and (60) it follows immediately that

$$\frac{\mathcal{B}_{-1}^2 Q_n + 2\mathcal{B}_0 Q_n}{Q_n} = \frac{\mathcal{B}_{-1}^2 \tilde{Q}_n + 2\mathcal{B}_0 \tilde{Q}_n}{\tilde{Q}_n} = \frac{\mathcal{B}_{-1}^2 P_n - 2\mathcal{B}_0 P_n}{P_n} = \frac{\mathcal{B}_{-1}^2 \tilde{P}_n - 2\mathcal{B}_0 \tilde{P}_n}{\tilde{P}_n} = 4(n - 2Q_n P_n). \tag{62}$$

From equations (62) one can readily obtain by simple algebra:

$$\mathcal{B}_{-1}(P_n \mathcal{B}_{-1} Q_n - Q_n \mathcal{B}_{-1} P_n) = -2\mathcal{B}_0(Q_n P_n) \tag{63}$$

as well as the identical equation for the corresponding "tilde"-quantities. Since the right-hand side of equation (63) is equal to the one for "tilde"-quantities, the corresponding left-hand

sides are equal too. Now we can apply the same argument as was used in Ref. 17 to obtain the crucial first integral (50) in this finite n Hermite (Gaussian) case, this time for the function

$$F_n = P_n \mathcal{B}_{-1} Q_n - Q_n \mathcal{B}_{-1} P_n = \tilde{P}_n \mathcal{B}_{-1} \tilde{Q}_n - \tilde{Q}_n \mathcal{B}_{-1} \tilde{P}_n. \tag{64}$$

The argument consists in that the last equality in (64) is true because clearly both sides tend to zero as all their arguments a_i tend to ∞, and by (63) and (62) both are translationally invariant w.r.t. any vector of the form $(a, a, ..., a)$. Then by (62) we can rewrite (64) as

$$\mathcal{B}_{-1} \ln \tilde{Q}_n - \mathcal{B}_{-1} \ln \tilde{P}_n = \mathcal{B}_{-1} \ln Q_n - \mathcal{B}_{-1} \ln P_n \tag{65}$$

Besides, by (62) we also have

$$\mathcal{B}_{-1} \ln \tilde{Q}_n + \mathcal{B}_{-1} \ln \tilde{P}_n = \mathcal{B}_{-1} \ln Q_n + \mathcal{B}_{-1} \ln P_n,$$

and so the simple relations emerge,

$$\mathcal{B}_{-1} \ln \tilde{Q}_n = \mathcal{B}_{-1} \ln Q_n, \tag{66}$$

$$\mathcal{B}_{-1} \ln \tilde{P}_n = \mathcal{B}_{-1} \ln Q_n. \tag{67}$$

Now by applying the above argument[17] once again, the announced result $\tilde{Q}_n = C_n Q_n$, $\tilde{P}_n = Q_n/C_n$ with some constant C_n follows. The constant C_n is possible to determine from the recursion, by definition of τ-ratios Q_n, P_n:

$$P_n = \frac{1}{Q_{n-1}}$$

and the limit when J becomes the whole real line. In this limit both u and w tend to zero, and the limit of $\tilde{Q}_n \tilde{P}_n$ is just $n/2$. Then also $Q_n/Q_{n-1} = n/2$ (this is as it should be because $Q_n P_n = b_{n-1}^2$ in this limit) and, comparing with the well-known value of the size n GUE matrix integral over the whole space $\mathbb{R}^n$ (see, e.g., Ref. 14),

$$\tau_n(0) = \frac{\pi^{n/2}}{2^{n(n-1)/2}} \prod_{j=1}^{n-1} j!, \tag{68}$$

we find that

$$C_n = \frac{2^n}{n!} \sqrt{\frac{n}{2\pi}}. \tag{69}$$

On the other hand, equation (58) is also possible to integrate once by the same argument that in the limit of all $a_i \to \infty$ both functions v and $\mathcal{B}_{-1} \ln \tau_n$ tend to zero. So by translation invariance they must be equal:

$$v = \mathcal{B}_{-1} \ln \tau_n^J / 2. \tag{7}$$

VI. CONCLUSION

A clear connection is exposed between the structures of equations derived by different —
Tracy-Widom and Adler-Shiota-van Moerbeke — methods. The structure of universal equations arising from resolvent kernel analysis is identified with that of 1-Toda–AKNS integrable
hierarchy. We have shown how, in the Gaussian case, to express the auxiliary coordinates,
related to the resolvent kernel in the first approach, in terms of ratios of τ-functions, the
natural coordinates of the second approach. The last ratios have here also a simple meaning
of probabilities. The close relation between Toda lattice hierarchies and systems of orthogonal
functions is crucial for this connection.

Although our explicit results have been obtained for single Gaussian matrix case, everything we have done here can as well be implemented for other one-matrix Hermitian unitarily
invariant ensembles. Also it is well-known that the (more universal) equations for various
large n limits can be readily obtained from the ones for finite n (see, e.g., Ref. 17 or Ref.
3). Moreover, our results pave the way to finding some similar connections for the case of
coupled Hermitian matrices. Such relations then may facilitate the more difficult analysis of
probabilities for the spectrum of coupled random matrices, with many possible applications
to random growth processes and other nonstationary nonlinear dynamics phenomena. This
work is in progress.

VII. APPENDIX: J.HARNAD'S PROCEDURE – FINITE n GAUSSIAN CASE

Following notations of Harnad,[11] introduce new TW variables:

$$X_{2j} = 2iq_{2j}, \ X_{2j+1} = 2q_{2j+1}, \tag{A1}$$

$$Y_{2j} = ip_{2j}, \ Y_{2j+1} = p_{2j+1}. \tag{A2}$$

Then, in terms of these variables, the first derivatives of τ-functions are the Poisson commuting Hamiltonians G_j^H for a finite integrable dynamical system[11] with canonical coordinates
$\{X_j, Y_j, u, w\}$ in our case,

$$G_j^H = \frac{\partial \ln \tau_n^J}{\partial a_j} = (-1)^{j+1} R(a_j, a_j) =$$

$$= -a_j X_j Y_j + (\sqrt{2n} - 2u)Y_j^2 + 1/4(\sqrt{2n} + 2w)X_j^2 - 1/4 \sum_{k \neq j}^{2m} \frac{(X_j Y_k - Y_j X_k)^2}{a_j - a_k}, \qquad (A3)$$

the relevant TW equations are

$$\frac{\partial X_j}{\partial a_k} = -1/2 \frac{X_j Y_k - Y_j X_k}{a_j - a_k} X_k, \qquad (39')$$

$$\frac{\partial Y_j}{\partial a_k} = -1/2 \frac{X_j Y_k - Y_j X_k}{a_j - a_k} Y_k, \qquad (40')$$

for $j \neq k$ and

$$\frac{\partial u}{\partial a_k} = -1/4 X_k^2, \qquad (43')$$

$$\frac{\partial w}{\partial a_k} = -Y_k^2, \qquad (45')$$

$$\frac{\partial X_j}{\partial a_j} = -a_j X_j + 2(\sqrt{2n} - 2u)Y_j + 1/2 \sum_{k \neq j}^{2m} \frac{X_j Y_k - Y_j X_k}{a_j - a_k} X_k, \qquad (46)$$

$$\frac{\partial Y_j}{\partial a_j} = a_j Y_j - 1/2(\sqrt{2n} + 2w)X_j + 1/2 \sum_{k \neq j}^{2m} \frac{X_j Y_k - Y_j X_k}{a_j - a_k} Y_k. \qquad (47)$$

Let

$$R_k^H = \sum_{j=1}^{2m} a_j^k G_j^H, \qquad (A4)$$

$$Q_k = \sum_{j=1}^{2m} a_j^k X_j^2, \; P_k = \sum_{j=1}^{2m} a_j^k Y_j^2, \; S_k = \sum_{j=1}^{2m} a_j^k X_j Y_j. \qquad (A5)$$

Then the first integral (50) is

$$2\sqrt{2n}(u - w) + 4uw - S_0 = 0. \qquad (A6)$$

We express by the above formulas (let $\tilde{u} = \sqrt{2n} - 2u$, $\tilde{w} = \sqrt{2n} + 2w$):

$$\mathcal{B}_{-1} \ln \tau_n^J = R_0^H = \sum_{j=1}^{2m} G_j^H = -S_1 + \tilde{u}P_0 + 1/4\tilde{w}Q_0, \qquad (A7)$$

$$\mathcal{B}_0 \ln \tau_n^J = R_1^H = \sum_{j=1}^{2m} a_j G_j^H = -S_2 + \tilde{u}P_1 + 1/4\tilde{w}Q_1 - 1/8\sum_{j,k}(X_jY_k - Y_jX_k)^2 =$$

$$= -S_2 + \tilde{u}P_1 + 1/4\tilde{w}Q_1 - 1/4Q_0P_0 + 1/4S_0^2, \qquad (A8)$$

we need also

$$\mathcal{B}_1 \ln \tau_n^J = R_2^H = \sum_{j=1}^{2m} a_j^2 G_j^H = -S_3 + \tilde{u}P_2 + 1/4\tilde{w}Q_2 - 1/8\sum_{j,k}(a_j + a_k)(X_jY_k - Y_jX_k)^2 =$$

$$= -S_3 + \tilde{u}P_2 + 1/4\tilde{w}Q_2 - 1/4(Q_0P_1 + Q_1P_0) + 1/2S_0S_1, \qquad (A9)$$

because $\sum_{j,k}(a_j + a_k)(X_jY_k - Y_jX_k)^2 = 2\sum_{j,k} a_j(X_j^2Y_k^2 + Y_j^2X_k^2 - 2X_jX_kY_jY_k) = 2(Q_1P_0 + Q_0P_1 - 2S_0S_1)$. We now find from (A7)–(A9) all the terms appearing in the ASvM boundary-KP equation (34). Applying the operators $\mathcal{B}_k$ to (A7)–(A9), we use the Poisson commutativity[11] of $\{G_j^H\}_{j=1,\dots,2m}$. This property means that the derivatives of R_j^H with respect to the a_j's are just given by their *explicit* dependence on these parameters.[11] This greatly simplifies the necessary differentiations so we obtain:

$$\mathcal{B}_{-1}^2 \ln \tau_n^J = \mathcal{B}_{-1}R_0^H = -S_0, \qquad (A10)$$

$$\mathcal{B}_{-1}\mathcal{B}_1 \ln \tau_n^J = \mathcal{B}_{-1}R_2^H = -3S_2 + 2\tilde{u}P_1 + 1/2\tilde{w}Q_1 - 1/2Q_0P_0 + 1/2S_0^2, \qquad (A11)$$

$$\mathcal{B}_0^2 \ln \tau_n^J = \mathcal{B}_0 R_1^H = -2S_2 + \tilde{u}P_1 + 1/4\tilde{w}Q_1, \qquad (A12)$$

$$\mathcal{B}_{-1}^4 \ln \tau_n^J = \mathcal{B}_{-1}^2(\mathcal{B}_{-1}^2 \ln \tau_n^J) = \mathcal{B}_{-1}^2(B_{-1}R_0^H) = -\mathcal{B}_{-1}^2 S_0 = -\sum_i \frac{\partial}{\partial a_i} \sum_l \frac{\partial}{\partial a_l} \sum_j X_jY_j =$$

$$= \sum_i \frac{\partial}{\partial a_i} \sum_j [2\tilde{u}Y_j^2 - 1/2\tilde{w}X_j^2] =$$

$$= -\sum_j [2\tilde{u}\cdot 2Y_j(a_jY_j - \tilde{w}X_j/2) - \tilde{w}/2\cdot 2X_j(-a_jX_j + 2\tilde{u}Y_j)] - \sum_j [4Y_j^2\cdot\sum_i X_i^2/4 + X_j^2\cdot\sum_i Y_i^2] =$$

$$= \sum_j [-2\sum_i X_i^2Y_j^2 - 4a_j\tilde{u}Y_j^2 - a_j\tilde{w}X_j^2 + 4\tilde{u}\tilde{w}X_jY_j] =$$

$$= -2Q_0P_0 - 4\tilde{u}P_1 - \tilde{w}Q_1 + 4\tilde{u}\tilde{w}S_0. \qquad (A13)$$

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Now we substitute the expressions (A8), (A10)–(A13) into the left-hand side of equation (34) and obtain

$$(\mathcal{B}_{-1}^4 + 8n\mathcal{B}_{-1}^2 + 12\mathcal{B}_0^2 + 24\mathcal{B}_0 - 16\mathcal{B}_{-1}\mathcal{B}_1)\ln\tau_n + 6(\mathcal{B}_{-1}^2\ln\tau_n)^2 =$$

$$= [-2Q_0P_0 - 4\tilde{u}P_1 - \tilde{w}Q_1 + 4\tilde{u}\tilde{w}S_0] - 8nS_0 + 12[-2S_2 + \tilde{u}P_1 + 1/4\tilde{w}Q_1] +$$

$$24[-S_2 + \tilde{u}P_1 + 1/4\tilde{w}Q_1 - 1/4Q_0P_0 + 1/4S_0^2] - 16[-3S_2 + 2\tilde{u}P_1 + 1/2\tilde{w}Q_1 - 1/2Q_0P_0 + 1/2S_0^2] + 6S_0^2$$

$$= 4S_0[(\sqrt{2n} - 2u)(\sqrt{2n} + 2w) - 2n + S_0] = 4S_0[-2\sqrt{2n}(u - w) - 4uw + S_0] = 0,$$

by the TW first integral (A6). Thus, the ASvM boundary-KP equation for the finite n Gaussian case (34) is proved by expressing the derivatives of the τ_n^J-function in terms of the TW variables, in the way of Ref. 11.

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